Exam Algebraic Structures, Wednesday April 8th 2015, 9.00–12.00.
(Possible points: 40, including 4 for free.)

(1) A consequence of a theorem in *algebraic number theory* is, that in the ring $\mathbb{Z}[\sqrt{-10}]$, for every ideal I an $n > 0$ exists such that I^n is a principal ideal.
  (a) [2 points] Find such n for the ideal $I = (2, \sqrt{-10})$.
  (b) [2 points] Is $\mathbb{Z}[\sqrt{-10}]$ a principal ideal domain?
  (c) [2 punten] Show that in the ring $\mathbb{Z}[\sqrt{-11}]$ the ideal $J = (2, 1+\sqrt{-11})$ satisfies $J^2 = (2) \cdot J$.
  (d) [1point] Show that for the ideal J in (c) and for all $n > 0$, one has $J^n = (2^{n-1}) \cdot J$. Show that as a consequence, every $a \in J^n$ can be expressed as $a = 2^{n-1}b$ for some $b \in J$.
  (e) [1 point] Show that J (defined in (c)) is not a principal ideal.
  (f) [2 points] Prove that no positive power of the ideal J defined in (c) is principal. Hint: were $J^n = (a)$, then $a = 2^{n-1}b$, in which b generates the ideal J, a contradiction...
  (g) [2 points] Is $\mathbb{Z}[\sqrt{-11}]$ factorial? (Consider for example the irreducibility of 2 and the primality of the ideal (2).)

(2) In this problem K is a field.
  (a) [2 points] Prove that $x^4 + x^2 + 1 \in K[x]$ is not a unit and is not irreducible.
  (b) [3 points] Take a monic irreducible polynomial $f(x) \in K[x]$, write $d = \text{degree}(f)$, and consider $g(x) := f(x^2)$. Explain that if $\alpha \in \Omega_K^g$ (the splitting field of g over K) satisfies $g(\alpha) = 0$, then $[K[\alpha] : K] \in \{d, 2d\}$.
  (c) [2 points] With notations as in (b), show that if $\text{Char}(K) = 0$ and $f(x) \neq x$, then $g(x)$ has no multiple zeros in Ω_K^g.
  (d) [2 points] Again with notations as in (b), prove that if $h(x) \in K[x]$ divides $g(x)$, then so does $h(-x) \in K[x]$.
  (e) [3 points] Again with notations as in (b), prove that if $\text{Char}(K) = 0$ and $g(x) \in K[x]$ is reducible, then a monic monic irreducibele $h(x) \in K[x]$ exists such that $g(x) = (-1)^d h(x)h(-x)$. You can do this in three steps: first show that a monic irreducible divisor $h(x)$ must have degree d. Next show that $h(x) \neq h(-x)$, since otherwise f would be reducible. Conclusion: either $h(x) = -h(-x)$, or $h(x) \neq \pm h(-x)$. In both cases the result follows.

(3) This exercise discusses the polynomial $f = x^6 + 4x^4 + 2x^3 + 5x^2 + 5$, which has in $\mathbb{Z}[i][x]$ the factor $x^3 + (2+i)x + 1 - 2i$.
  (a) [3points] Explain that the presented divisor of f in $\mathbb{Z}[i][x]$ is an Eisenstein polynomial, hence an irreducible polynomial in $\mathbb{Z}[i][x]$.
  (b) [2 points] Split f in irreducible factors in the ring $\mathbb{Z}[i][x]$.
  (c) [2 points] Use a zero α of f and the fields $\mathbb{Q} \subset \mathbb{Q}(i) \subset \mathbb{Q}(i, \alpha)$ to show that $f \in \mathbb{Z}[x]$ is irreducible.
  (d) [2 points] Give an irreducible factor of $f \bmod 17 \in \mathbb{F}_{17}[x]$.
  (e) [3 points] Show that if p is a prime number that is not irreducible in $\mathbb{Z}[i]$, then $x^2 + 1 \in \mathbb{F}_p[x]$ splits and therefore as a consequence $f \bmod p \in \mathbb{F}_p[x]$ is not irreducible.